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A multiquadric quasi-interpolation with linear reproducing and preserving monotonicity☆

Ronghua Chen [a,b,*], Xuli Han [a], Zongmin Wu [c]

[a] School of Mathematical Science and Computing Technology, Central South University, Changsha, Hunan, 410083, PR China

[b] School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan, 411201, PR China

[c] Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematics, Fudan University, Shanghai, 200433, PR China

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ABSTRACT

In this paper, we develop a multiquadric (MQ) quasi-interpolation which has the properties of linear reproducing and preserving monotonicity. Moreover, we give its approximation error by theoretic analysis and illustrate the effect by means of two examples. One of the examples is to approach the linear combination of two sine functions with different frequencies. Another is to approximate a function with discontinuity. From the results of the examples, we believe that the present MQ quasi-interpolation is feasible.

1. Introduction

Since Hardy proposed it in 1968 [1], the multiquadric (MQ) which is a kind of radial basis function (RBF) has been investigated thoroughly. Hardy [2] summarized the achievement of study of MQ from 1968 to 1988 and showed that MQ can be applied to hydrology, geodesy, photogrammetry, surveying and mapping, geophysics and crustal movement, geology and mining and so on. Since Kansa [3,4] successfully modified MQ for solving partial differential equation (PDE), more and more researchers have been attracted by this meshless, scattered data approximation scheme (see, for example, [5–21]).

In Franke's review paper [22], the MQ was rated one of the best methods among 29 scattered data interpolation schemes based on their accuracy, stability, efficiency, memory requirement, and ease of implementation.

So far, there are four kinds of MQ quasi-interpolation, namely, $\mathcal{L}_A$, $\mathcal{L}_B$, and $\mathcal{L}_C$ developed in [23] and $\mathcal{L}_D$ in [24].

Based on Wu and Schaback's paper, Ling proposed a multilevel univariate quasi-interpolation scheme, this scheme can converge with a rate of $O(h^{2.5}\log h)$ as $c = O(h)$ [25].

In this paper, we devise a new MQ quasi-interpolation which has the properties of linear reproducing and preserving monotonicity. Moreover, we give its approximation error and two examples. From the results of the examples, we believe that the present MQ quasi-interpolation possesses almost second order accuracy.

The rest of this paper is organized as follows. In Section 2, we introduce the univariate MQ quasi-interpolation and its properties. In Section 3, we give two examples. One is to quasi-interpolate a C^∞ function and another for a discontinuous function. The results are also acceptable. So the technique is valid. In Section 4, we derive conclusions.

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* Corresponding author at: School of Mathematics and Computational Science, Hunan University of Science and Technology, Xiangtan, Hunan, 411201, PR China.

E-mail addresses: rhchen05@126.com (R. Chen), xlhan@mail.csu.edu.cn (X. Han), zmwu@fudan.edu.cn (Z. Wu).

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The main results can be found in Refs. [26–28]. But, in [27,28], we don't give the proof for most of the theorems. Again the proof given in [26] are based on two conditions, i.e. periodic function and inflow-outflow boundary conditions. As a matter of fact, the conclusion in [26] can be extended. Furthermore, in [26–28], we use the univariate MQ quasi-interpolation to solve differential equations but don't illustrate the approximative effect. In addition to these, we don't show the relation between the quasi-interpolation given in [26–28] and $\mathcal{L}_D$. Therefore we can say that the work in this paper is the continuation and deepening of work in [26–28]. That is just the reason as to why we have written this paper.

2. Multiquadric quasi-interpolation

In 1992, Beatson and Powell [23] proposed three univariate multiquadric quasi-interpolations, namely, $\mathcal{L}_{\mathcal{A}}$, $\mathcal{L}_{\mathcal{B}}$, and $\mathcal{L}_{\mathcal{C}}$, to approximate a function $\{f(x),\ x_0 \le x \le x_n\}$ from the space that is spanned by the multiquadrics $\{\phi_j(x) = \sqrt{(x-x_j)^2+c^2},\ x \in \mathbb{R},\ j = 0, \dots, n\}$ and linear function, where c is a positive constant and the centers $\{x_j : j = 0, \dots, n\}$ being given distinct points in the interval $[x_0, x_n]$, in which $\mathcal{L}_B$ is constant reproducing, $\mathcal{L}_C$ is linear reproducing; while $\mathcal{L}_A$ is not constant reproducing, $\mathcal{L}_B$ is not linear reproducing. Afterward, Beatson and Dyn [29] have studied the properties of the Ψ-splines, the combination of the MQs, and obtained the error estimates for quasi-interpolation schemes involving MQ based on a finite number of centers.

Wu and Schaback [24] have proposed the univariate multiquadric quasi-interpolation $\mathcal{L}_{\mathcal{D}}$ on $[x_0, x_n]$ and proven that the scheme is shape preserving and convergent.

In this section, we construct a kind of special MQ quasi-interpolation which is the generalization of $\mathcal{L}_{\mathcal{D}}$.

Given points $\{(x_j, f_j)\}_{j=0}^n$, where $x_0 < x_1 < \cdots < x_n$, and $f_j = f(x_j)$, we construct the univariate quasi-interpolation in the form of

$$f^*(x) = \sum_{j=0}^{n} f_j \psi_j(x), \tag{2.1}$$

where, $\psi_j(x)$ $(j = 0, 1, \dots, n)$ are some selected functions.

Now, for the sake of ease of reading, we introduce some definitions related to quasi-interpolation.

Definition 2.1. If the quasi-interpolation $f^*(x)$ possesses the property

$$f^*(x) \equiv C \quad \text{if } f_0 = f_1 = \cdots = f_n = C, \tag{2.2}$$

where C is any real constant, we say that the quasi-interpolation is **constant reproducing** on $[x_0, x_n]$.

Definition 2.2. We say that the quasi-interpolation $f^*(x)$ **possesses linear reproducing property** on $[x_0, x_n]$, if $f^*(x) = px + q$ as $f_j = px_j + q, j = 0, \dots, n$, for all $p, q \in \mathbb{R}$.

Remark 2.1. It is obvious that if a quasi-interpolation $f^*(x)$ possesses linear reproducing property on $[x_0, x_n]$ then it must be constantly reproducing.

Definition 2.3. If the quasi-interpolation $f^*(x)$ is monotone increasing (decreasing) for monotone increasing (decreasing) data $f_j, j = 0, \dots, n$, then we say that it **possesses preserving monotonicity** on $[x_0, x_n]$.

Let's recall $\mathcal{L}_{\mathcal{D}}$ [24]. The $\mathcal{L}_{\mathcal{D}}$ defined by Wu and Schaback is as follows:

$$(\mathcal{L}_{\mathcal{D}} f)(x) = f_0\alpha_0(x) + f_1\alpha_1(x) + \sum_{j=2}^{n-2} f_j\psi_j(x) + f_{n-1}\alpha_{n-1}(x) + f_n\alpha_n(x) \tag{2.3}$$

where,

$$\begin{aligned}
\alpha_0(x) &= \frac{1}{2} + \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},\\
\alpha_1(x) &= \frac{\phi_2(x) - \phi_1(x)}{2(x_2 - x_1)} - \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)},\\
\alpha_{n-1}(x) &= \frac{(x_n - x) - \phi_{n-1}(x)}{2(x_n - x_{n-1})} - \frac{\phi_{n-1}(x) - \phi_{n-2}(x)}{2(x_{n-1} - x_{n-2})},\\
\alpha_n(x) &= \frac{1}{2} + \frac{\phi_{n-1}(x) - (x_n - x)}{2(x_n - x_{n-1})},\\
\phi_j(x) &= \sqrt{(x - x_j)^2 + c^2}, \quad j = 1, \dots, n-1,\ c \in \mathbb{R},\\
\psi_j(x) &= \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1} - x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}, \quad j = 2, \dots, n-2.
\end{aligned}$$

The formula (2.3) can be rewritten as:

$$(\mathcal{L}_{\mathcal{D}}f)(x) = \frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{f_{j+1}-f_j}{x_{j+1}-x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}}\right)\phi_j(x) + \frac{f_0+f_n}{2} + \frac{f_1-f_0}{2(x_1-x_0)}(x-x_0) - \frac{f_n - f_{n-1}}{2(x_n - x_{n-1})}(x_n - x). \tag{2.4}$$

Let

$$\phi_{-1}(x) = |x - x_{-1}|, \phi_0(x) = |x-x_0|, \phi_n(x) = |x - x_n|, \phi_{n+1}(x) = |x - x_{n+1}|, \tag{2.5}$$

and restrict $x \in [x_0, x_n]$, then $(\mathcal{L}_{\mathcal{D}}f)(x)$ can be rewritten as:

$$(\mathcal{L}_{\mathcal{D}}f)(x) = \sum_{j=0}^{n} f_j \psi_j(x) \tag{2.6}$$

where,

$$\psi_j(x) = \frac{\phi_{j+1}(x) - \phi_j(x)}{2(x_{j+1}-x_j)} - \frac{\phi_j(x) - \phi_{j-1}(x)}{2(x_j - x_{j-1})}, \quad j = 0, 1, \ldots, n. \tag{2.7}$$

$x_{-1} < x_0, x_{n+1} > x_n$. So that, in this paper, we consider the univariate quasi-interpolation in form of (2.1) and (2.7) mainly, where $\phi_{-1}(x), \phi_0(x), \phi_n(x), \phi_{n+1}(x)$ can be different from (2.5).

We can obtain a quasi-interpolation $f^*(x)$ which possesses the following properties.

Theorem 2.1. *If*

$$\phi_{-1}(x) = \phi_0(x) + x_0 - x_{-1}, \tag{2.8}$$
$$\phi_n(x) = \phi_0(x) - 2x + x_0 + x_n, \tag{2.9}$$
$$\phi_{n+1}(x) = \phi_n(x) + x_{n+1} - x_n, \tag{2.10}$$

then the quasi-interpolation $f^*(x)$ defined by formulae (2.1) and (2.7) possesses the property of linear reproducing on $[x_0, x_n]$. Furthermore, on $[x_0, x_n]$, $f^*(x)$ can be written as three equivalent forms as follows:

$$\begin{aligned} f^*(x) &= \frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{\phi_{j+1}(x)-\phi_j(x)}{x_{j+1}-x_j} - \frac{\phi_j(x)-\phi_{j-1}(x)}{x_j - x_{j-1}}\right) f_j \\ &\quad + \frac{1}{2}\left(1 + \frac{\phi_1(x)-\phi_0(x)}{x_1 - x_0}\right) f_0 + \frac{1}{2}\left(1 - \frac{\phi_n(x) - \phi_{n-1}(x)}{x_n - x_{n-1}}\right) f_n; \end{aligned} \tag{2.11}$$

$$f^*(x) = \frac{f_0 + f_n}{2} + \frac{1}{2}\sum_{j=0}^{n-1}\frac{\phi_j(x) - \phi_{j+1}(x)}{x_{j+1}-x_j}(f_{j+1}-f_j); \tag{2.12}$$

$$f^*(x) = \frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{f_{j+1}-f_j}{x_{j+1}-x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}}\right)\phi_j(x) + \frac{f_0+f_n}{2} + \frac{f_1 - f_0}{2(x_1-x_0)}\phi_0(x) - \frac{f_n - f_{n-1}}{2(x_n - x_{n-1})}\phi_n(x). \tag{2.13}$$

Moreover, on $[x_0, x_n]$, we have

$$(f^*(x))' = \frac{1}{2}\sum_{j=0}^{n-1}\frac{\phi_j'(x) - \phi_{j+1}'(x)}{x_{j+1}-x_j}(f_{j+1}-f_j) \tag{2.14}$$

and

$$(f^*(x))'' = \frac{1}{2}\sum_{j=0}^{n-1}\frac{\phi_j''(x) - \phi_{j+1}''(x)}{x_{j+1}-x_j}(f_{j+1}-f_j) \tag{2.15}$$

Again, in general, we have

$$(f^*(x))^{(k)} = \frac{1}{2}\sum_{j=0}^{n-1}\frac{\phi_j^{(k)}(x) - \phi_{j+1}^{(k)}(x)}{x_{j+1}-x_j}(f_{j+1}-f_j) \tag{2.16}$$

where, $x_{-1} < x_0$ and $x_{n+1} > x_n$.

Proof. First, due to (2.1) and (2.7), we obtain

$$\begin{aligned} f^*(x) &= \frac{1}{2}\sum_{j=0}^{n}\left(\frac{\phi_{j+1}(x)-\phi_j(x)}{x_{j+1}-x_j}-\frac{\phi_j(x)-\phi_{j-1}(x)}{x_j-x_{j-1}}\right)f_j \\ &= \frac{1}{2}\sum_{j=0}^{n}\frac{\phi_{j+1}(x)-\phi_j(x)}{x_{j+1}-x_j}f_j-\frac{1}{2}\sum_{j=-1}^{n-1}\frac{\phi_{j+1}(x)-\phi_j(x)}{x_{j+1}-x_j}f_{j+1} \\ &= f_0\frac{\phi_{-1}(x)-\phi_0(x)}{2(x_0-x_{-1})}+f_n\frac{\phi_{n+1}(x)-\phi_n(x)}{2(x_{n+1}-x_n)}+\frac{1}{2}\sum_{j=0}^{n-1}\frac{\phi_j(x)-\phi_{j+1}(x)}{x_{j+1}-x_j}(f_{j+1}-f_j) \end{aligned}$$

then, by (2.8) and (2.10), we have (2.12). Formulae (2.11)–(2.13) are equivalent each other by calculating straightly. Differentiating (2.12) with respect to x, we have formulae (2.14)–(2.16). Quasi-interpolation $f^*(x)$ possesses the property of linear reproducing, if we can prove:

$$f^*(x) = 1, \quad \text{for } f(x) = 1, \tag{2.17}$$

and

$$f^*(x) = x, \quad \text{for } f(x) = x. \tag{2.18}$$

Due to (2.13), we can see (2.17) is valid. From (2.13) and (2.9), we have (2.18). □

Remark 2.2. From the proof of the above theorem, we can see that it does not require the condition (2.9) for the validity of formulae (2.11)–(2.16), and the property of constant reproducing.

For $\mathcal{L}_{\mathcal{D}}$, noting that (2.5), we have the following corollary.

Corollary 2.1. *The univariate multiquadric quasi-interpolation $\mathcal{L}_{\mathcal{D}}$ possesses linear reproducing on $[x_0, x_n]$.*

Proof. Due to (2.5) and $x \in [x_0, x_n]$, then (2.8)–(2.10) are valid. □

Definition 2.4. The quasi-interpolation $f^*(x)$ defined by formulae (2.1) and (2.7)–(2.10) and

$$\phi_j(x) = \sqrt{(x-x_j)^2+c^2}, \quad c > 0,\ c \in \mathbb{R},\ j = 0, 1, \ldots, n-1 \tag{2.19}$$

is just the MQ quasi-interpolation developed in this paper.

Letting $c = 0$ for $\phi_0(x)$ in 2.4, then $f^*(x)$ is just $\mathcal{L}_{\mathcal{D}}$. So we believe the quasi-interpolation defined by Definition 2.4 is the generalization of $\mathcal{L}_{\mathcal{D}}$. It has also the properties as follows.

Theorem 2.2. *The MQ quasi-interpolation $f^*(x)$ defined by formulae (2.1), (2.7)–(2.10) and (2.19) has the properties of linear reproducing and preserving monotonicity on $[x_0, x_n]$. Moreover, on $[x_0, x_n]$, we have (2.11)–(2.16).*

Proof. Due to Theorem 2.1, we only have to prove the MQ quasi-interpolation $f^*(x)$ defined by Definition 2.4 has the property preserving monotonicity on $[x_0, x_n]$.

Wu and Schaback [24] have proven:

$$-1 \le \phi_j'(x) \le \phi_{j-1}'(x) \le 1 \quad \text{for } j = 1, \ldots, n-1.$$

Again, from (2.9), we have

$$\phi_n'(x) = \phi_0'(x) - 2 \le -1 \le \phi_{n-1}'(x).$$

So,

$$\phi_j'(x) - \phi_{j+1}'(x) \ge 0, \quad \text{for all } j = 0, 1, \ldots, n-1.$$

Then, from (2.14), we complete the proof. □

Theorem 2.3. *Let*

$$h = \max_{1\le j\le n}\{x_j - x_{j-1}\}.$$

For any real number $c > 0$, $x \in [x_0, x_n]$ and function $f(x) \in C^2(x_0, x_n)$, the MQ quasi-interpolation $f^*(x)$ defined by Definition 2.4 satisfies:

$$\|f^*(x) - f(x)\|_\infty \le k_0 C_h + k_1 h^2 + k_2 ch + k_3 c^2 \log h \tag{2.20}$$

where,

$$C_h = \min\left\{c, \frac{c^2}{h}\right\},$$

k_0, k_1, k_2, and k_3 are constants independent of h and c.

Proof. Due to (2.13), we have

$$\begin{aligned} 2f^*(x) &= f_0 + f_n + \phi_0(x)\frac{f_1 - f_0}{x_1 - x_0} - \phi_n(x)\frac{f_n - f_{n-1}}{x_n - x_{n-1}} + \sum_{j=1}^{n-1}\phi_j(x)\left(\frac{f_{j+1} - f_j}{x_{j+1} - x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}}\right) \\ &= f_0 + f_n + \phi_0(x)\Delta^1(x_0, x_1)f - \phi_n(x)\Delta^1(x_{n-1}, x_n)f + \sum_{j=1}^{n-1}[\phi_j(x)(x_{j+1} - x_{j-1})\Delta^2(x_{j-1}, x_j, x_{j+1})f]. \end{aligned}$$

where, Δ^1 and Δ^2 is the first and second divided difference respectively. Since $f(x) \in C^2(x_0, x_n)$, the first and second divided difference are bounded. Denoting the piecewise linear interpolation by $L(x)$ whose known initial data are same as that of constructing $f^*(x)$, then we have

$$2(f^*(x) - L(x)) = M(x) + N(x).$$

In which,

$$M(x) = \sum_{j=1}^{n-1}\{(\phi_j(x) - |x - x_j|)(x_{j+1} - x_{j-1})\Delta^2(x_{j-1}, x_j, x_{j+1})f\},$$

$$\begin{aligned} N(x) &= (\phi_0(x) - (x - x_0))\Delta^1(x_0, x_1)f - (\phi_n(x) - (x_n - x))\Delta^1(x_{n-1}, x_n)f \\ &= (\phi_0(x) - (x - x_0))(\Delta^1(x_0, x_1)f - \Delta^1(x_{n-1}, x_n)f). \end{aligned} \tag{2.21}$$

So,

$$\|f^*(x) - f(x)\|_\infty \le \frac{1}{2}\|N(x)\|_\infty + \|\frac{1}{2}M(x) + L(x) - f(x)\|_\infty.$$

Following the idea developed in [24], we obtain:

$$\left\|\frac{1}{2}M(x) + L(x) - f(x)\right\|_\infty \le k_1h^2 + k_2ch + k_3c^2\log h,$$

and

$$\begin{aligned} &\sqrt{c^2 + y^2} - |y| \le c, \quad \text{for } c \ge 0,\ y \in \mathbb{R}, \\ &\sqrt{c^2 + y^2} - |y| \le \frac{c^2}{2|y|}, \quad \text{for } c, y \in \mathbb{R},\ y \ne 0. \end{aligned}$$

Therefore, we get

$$\frac{1}{2}\|N(x)\|_\infty \le k_0C_h.$$

That is to say, (2.20) is valid. □

Remark 2.3. In the proof of the above theorem, we have used

$$\begin{aligned} L(x) &= \frac{1}{2}\sum_{j=0}^{n}\left(\frac{|x - x_{j+1}| - |x - x_j|}{x_{j+1} - x_j} - \frac{|x - x_j| - |x - x_{j-1}|}{x_j - x_{j-1}}\right)f_j \\ &= \frac{|x - x_{-1}| - |x - x_0|}{2(x_0 - x_{-1})}f_0 + \frac{|x - x_{n+1}| - |x - x_n|}{2(x_{n+1} - x_n)}f_n + \frac{f_1 - f_0}{2(x_1 - x_0)}|x - x_0| \\ &\quad - \frac{f_n - f_{n-1}}{2(x_n - x_{n-1})}|x - x_n| + \frac{1}{2}\sum_{j=1}^{n-1}\left(\frac{f_{j+1} - f_j}{x_{j+1} - x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}}\right)|x - x_j|. \end{aligned}$$

So, as $x \in [x_0, x_n]$, we have

Table 1
Results of MQ quasi-interpolation approximating $f(x) = \sin 2\pi x + 0.5 \sin \pi x$ as $c = 0.001$.

n	L_∞-norm	r_c	L_1-norm	r_c
20	0.051238		0.020309	
40	0.012844	1.9916	0.0051486	1.9798
80	3.2992×10^{-3}	1.9654	1.3593×10^{-3}	1.9214
160	8.8823×10^{-4}	1.8931	3.9728×10^{-4}	1.7746

Table 2
Results of MQ quasi-interpolation approximating $f(x) = \sin 2\pi x + 0.5 \sin \pi x$ as $c = h^2$.

	L_∞-norm	r_c	L_1-norm	r_c
20	0.055689		0.025055	
40	0.013162	2.0810	0.0055231	2.1815
80	3.2525×10^{-3}	2.0168	1.3085×10^{-3}	2.0776
160	7.9994×10^{-4}	2.0236	3.1875×10^{-4}	2.0374

$$L(x) = \frac{f_0 + f_n}{2} + \frac{x - x_0}{2}\Delta^1(x_0, x_1)f - \frac{x_n - x}{2}\Delta^1(x_{n-1}, x_n)f + \frac{1}{2}\sum_{j=1}^{n-1}|x - x_j|(x_{j+1} - x_{j-1})\Delta^2(x_{j-1}, x_j, x_{j+1})f.$$

From Theorem 2.3, we note that the accuracy of the MQ quasi-interpolation $f^*(x)$ defined by Definition 2.4 is $\mathcal{O}(h^2)$ if $c = \mathcal{O}(h^2)$.

Corollary 2.2. *Let*

$$h = \max_{1 \le j \le n}\{x_j - x_{j-1}\}.$$

For any real number $c > 0$, $x \in [x_0, x_n]$ and function $f(x) \in C^2(x_0, x_n)$, the univariate multiquadric quasi-interpolation $\mathcal{L}_\mathcal{D}$ satisfies:

$$\|(\mathcal{L}_\mathcal{D} f)(x) - f(x)\|_\infty \le k_1 h^2 + k_2 ch + k_3 c^2 \log h, \tag{2.22}$$

where, k_1, k_2, and k_3 are constants independent of h and c.

Proof. For $\mathcal{L}_\mathcal{D}$, due to $x \in [x_0, x_n]$ and (2.5), then $\phi_0(x) = x - x_0$, therefore $N(x) \equiv 0$. $\square$

3. Examples

In this section, we use the MQ quasi-interpolation $f^*(x)$ defined by Definition 2.4 to approximate two known functions. For the sake of simplicity, we take equidistant grids. We show the results by tables and figures. In the figures, MQQI is the abbreviation of MQ quasi-interpolation and "exact" represents the exact solution viz. the curve of the given function.

Now, we consider MQ quasi-interpolation to approximate the function $f(x) = \sin 2\pi x + 0.5 \sin \pi x$ which is a C^∞ function. For this function, we show the results by two tables and two figures. The tables include the L_∞ and L_1 norm of the error and their computational order of accuracy for some h, where $h = \frac{2}{n}$. The values of r_c in Tables 1 and 2 is the "computational order of accuracy" which is calculated by assuming the error to be a constant times h^{r_c}; this definition is meaningful only for h sufficiently small. The figures are the results taking $h = 0.05$, where, Fig. 1 for $c = 0.001$ and Fig. 2 for $c = h^2 = 0.0025$.

From Tables 1 and 2, we can say that the accuracy of the MQ quasi-interpolation is desirable. Especially, the accuracy reaches $\mathcal{O}(h^2)$ if we set $c = \mathcal{O}(h^2)$.

By Figs. 1 and 2, we can see that the MQ quasi-interpolation approaches the function very close, as $h = 0.05$ for $c = 0.001$ and $c = h^2 = 0.0025$.

Next, we consider the MQ quasi-interpolation of a discontinuous function as follows

$$f(x) = \begin{cases} \frac{10}{3}x, & 0 \le x \le 0.3, \\ 1, & 0.3 \le x \le 0.6, \\ 0, & 0.6 < x \le 1.0. \end{cases} \tag{3.1}$$

For $h = \frac{1}{n} = 0.01$ and $c = 0.001$, the results are shown in Fig. 3, where, the L_∞-norm and L_1-norm of the error vector is 0.0047508 and 1.0285×10^{-3}, respectively.

For $h = \frac{1}{n} = 0.01$ and $c = h^2 = 0.0001$, the results are shown in Fig. 4, where, the L_∞-norm and L_1-norm of the error vector is 4.975×10^{-3} and 1.0236×10^{-4}, respectively.

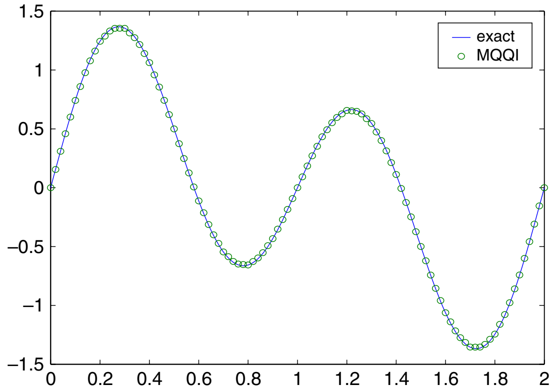

Fig. 1. Results of MQ quasi-interpolation as $h = 0.05$ and $c = 0.001$.

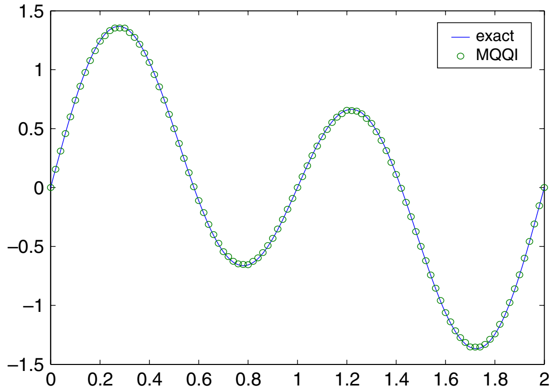

Fig. 2. Results of MQ quasi-interpolation as $h = 0.05$ and $c = h^2 = 0.0025$.

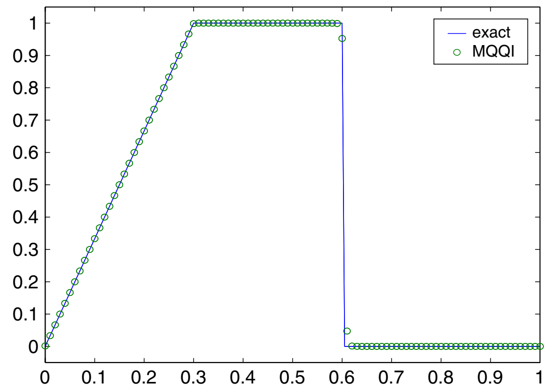

Fig. 3. Results of MQ quasi-interpolation for a discontinuous function as $h = 0.01$ and $c = 0.001$.

From Figs. 3 and 4, and the norm of error vector given above, we note that, for a discontinuous function, the shape parameter c influences the accuracy.

4. Conclusion

Summarily, the quasi-interpolation defined by formulae (2.1) and (2.7)–(2.10) possesses the property of linear reproducing. Again, (2.11)–(2.16) are valid. In addition to the properties above, the MQ quasi-interpolation defined by Definition 2.4 bears the property of preserving monotonicity and its error approximating a C^2 function can be described by (2.20).

For a smooth function, the accuracy of the MQ quasi-interpolation is desirable. In particular, the accuracy reaches $\mathcal{O}(h^2)$ if we set $c = \mathcal{O}(h^2)$.

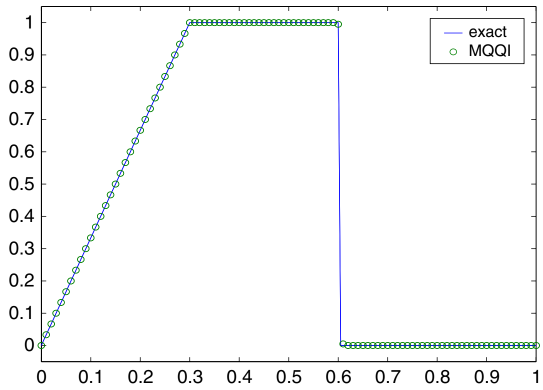

Fig. 4. Results of MQ quasi-interpolation for a discontinuous function as $h = 0.01$ and $c = h^2 = 0.0001$.

For a discontinuous function, there is a very close relation between the shape parameter c and the accuracy. As to the selection of the shape parameter, it is different from problem to problem. In general, we suggest $c = h^2$. From (2.4), we know that $\mathcal{L}_{\mathcal{D}}$ is convexity-preserving because of $c > 0$ and

$$\phi_j''(x) = \frac{c^2}{\sqrt{((x - x_j)^2 + c^2)^3}} > 0, \quad j = 1, \ldots, n - 1. \tag{4.1}$$

But, from (2.13) and (2.9), we have

$$(f^*(x))'' = \frac{1}{2} \sum_{j=1}^{n-1} \left(\frac{f_{j+1} - f_j}{x_{j+1} - x_j} - \frac{f_j - f_{j-1}}{x_j - x_{j-1}} \right) (\phi_j''(x) - \phi_0''(x)).$$

Therefor, we can't assert $f^*(x)$ is convexity-preserving although the results of our numerical experiments seem like that.

In terms of the technique given [25], the quasi-interpolation $f^*(x)$ in this paper can also become a multilevel univariate quasi-interpolation scheme.

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